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GENERALIZED TWO DIMENSIONAL CANONICAL TRANSFORM

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Abstract: *The two-dimensional canonical transform can be used in optical system analysis, image processing and pattern recognition. In this paper two-dimensional transform anonical is extended to the distribution of compact support. Analyticity theorem, inversion theorem, is proved for this transform. Lastly properties of kernel are discussed.*

Keyword:- *Canonical transform, two-dimensional canonical transform Generalized function, signal processing.*

I. Introduction:

Now a days fractional integral transform play an important role in signal processing, image reconstruction, pattern recognition, acoustic signal processing,[1], [2].The fractional Fourier transform[3], [4], which is the generalization of
the one-dimensional Fourier transforms is defined as

$$F^{\alpha}(s)=\sqrt{\frac{1-\cot\alpha}{2\pi}}e^{\frac{i}{2}\left(\cot\alpha . s^2\right)}\int_{-\infty}^{\infty}e^{-i\cot\alpha t^2}e^{\frac{i}{2}\cot\alpha t^2}f(t)dt......... \quad (1)$$

It has the following additivity property

$$o_F^{\beta}.o_F^{\alpha}=o_F^{\alpha+\beta} \qquad .... \quad (2)$$

In fact, the fractional Fourier transform is the special case of the canonical transform [5],[6]. The canonical transform is defined as

$$\therefore \left\{2DCT\ f(t,x)\right\}(s,w)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{-i\left(\frac{s}{b}t\right)}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t)dt \ \ b\neq 0.................(3)$$

The one-dimensional canonical transform can be extended in to two-dimensional canonical transform as follows.

$$\therefore \left\{2DCT\ f(t,x)\right\}(s,w)$$

$$=\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}e^{-i\left(\frac{s}{b}t\right)}e^{-i\left(\frac{w}{b}x\right)}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t,x)dxdt \qquad b\neq 0 \ \ ....(4)$$

Notation and terminology as per Zemanian [7].
This paper is organized as follows: Section 2 the definition two- dimensional canonical transform, and testing function space. Section 3 inversion and Analyticity theorem, are proved. Section 4 properties of kernel are discussed.

II. Definition two dimensional (2D) canonical transform:

Where we have, given the definition of two dimensional (2D) generalized canonical transform.

2.1 Two-dimensional Generalized canonical transform $E'(R\times R)$:

It can be easily proved that the functions $K_{f_1}(t,s)$ and $K_{f_2}(x,w)$ which are the functions of t and x are members of $E(R\times R)$.
where,

$$K_{f_1}(t,s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.e^{-i\left(\frac{s}{b}t\right)} \quad \text{and} \quad K_{f_2}(x,w)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}.e^{-i\left(\frac{w}{b}x\right)}$$

That is $\gamma_{E,k}\left\{K_{f_1}(t,s)K_{f_2}(x,w)\right\}=\sup_{\substack{-\infty<t<\infty \\ -\infty<x<\infty}}\left|D_t^k D_x^l K_{f_1}(t,s)K_{f_2}(x,w)\right|<\infty$

let $E^{'}(R\times R)$ denotes the dual of $E(R\times R)$. Therefore the generalized canonical transform of $f(t,x)\in E^{'}(R\times R)$ can be defined as

$$\left\{2DCT\ f(t,x)\right\}(s,w)=\left\langle f(t,x),K_{f_1}(t,s)K_{f_2}(x,w)\right\rangle$$

$\therefore\ \left\{2DCT\ f(t,x)\right\}(s,w)$

$$=\frac{1}{\sqrt{2\pi ib}}\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}e^{-i\left(\frac{s}{b}t\right)}.e^{-i\left(\frac{w}{b}x\right)}.e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f(t,x)\,dxdt$$

Where $K_{f_1}(t,s)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-i\left(\frac{s}{b}\right)t}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}$ when $b\neq 0$

$=\sqrt{d}e^{\frac{i}{2}\left(cds^2\right)}\delta(t-ds)$ when b = 0

$K_{f_2}(x,w)=\frac{1}{\sqrt{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}e^{-i\left(\frac{w}{b}\right)x}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}$ when $b\neq 0$

$=\sqrt{d}\,e^{\frac{i}{2}\left(cdw^2\right)}\delta(x-d.w)$ when b = 0

2.2 Definition of testing function space:

An infinitely differentiable complex valued function ϕ on R^n belongs to *E(R^n)*, if for each compact set. $I\subset s_a$, $J\subset s_b$ where $s_a=\left\{t:t\in R^n,\ \ |t|\le a,\ a>0\right\}$, $s_b=\left\{x:x\in R^n,\ \ |x|\le b,\ b>0\right\}$

and for $I\in R^n$, $J\in R^n$,

$\gamma_{E,k}\phi(t,x)=\sup_{\substack{-\infty<t<\infty \\ -\infty<t<\infty}}\left|D_t^k D_x^l\ \phi(t,x)\right|<\infty$ k=0,1,2,3..and *l*=0,1,2,3...

Thus $E\left(R^n\right)$ will denotes the space of all $\phi(t,x)\in E\left(R^n\right)$ with support contained in s_a and s_b. Note that space E is complete and a Frechet space, let $E^{'}$ denotes the dual space of E.

III. Inversion and Analyticity of Two Dimensional canonical transform:

3.1 Inverse of Two Dimensional canonical transform:

If $\left\{2DCT\ f(t,x)\right\}(s,w)$ is canonical transform of $f(t,x)$ then inverse of transform is given by

$$f(t,x)=\sqrt{\frac{2\pi i}{b}}\sqrt{\frac{2\pi i}{b}}e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{-i}{2}\left(\frac{a}{b}\right)x^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\left(\frac{s}{b}t\right)}e^{-i\left(\frac{w}{b}t\right)}e^{\frac{-i}{2}\left(\frac{d}{b}\right)s^2}e^{\frac{-i}{2}\left(\frac{d}{b}\right)w^2}\left\{2DCT\ f(t,x)\right\}(s,w)\,dsdw$$

3.2 Analyticity theorem:

Let $f\in E^1(R^n)$ and let its two canonical transform be defined by,

$$\left\{2DCT\ f(t,x)\right\}(s,w)=\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}e^{-i\left(\frac{s}{b}\right)t}e^{-i\left(\frac{w}{b}\right)x}f(t,x)\,dxdt$$

then $\left\{2DCT\ f(t,x)\right\}(s,w)$ is analytic on C^n,if the a, b, $\sup pf\subset s_a$ and s_b where $s_a=\left\{t:t\in R^n,\ |t|\le a,a>0\right\}$, $s_b=\left\{x:x\in R^n,\ |x|\le b,b>0\right\}$ moreover $\{2DCT(f,t)$ is differentiable and $D_s^kD_w^l\left\{2DCT\ f(t,x)\right\}(s,w)=\left\langle f(t,x),D_s^kD_w^lK_{f_1}(t,s)K_{f_2}(x,w)\right\rangle$

Proof: Let, $s:\{s_1,s_2,\ldots\ldots\ldots s_j\ldots\ldots s_n\}\in C^n$ and $w:\{w_1,w_2,\ldots\ldots\ldots w_j\ldots\ldots w_n\}\in C^n$

We first prove that, $\frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j}\{2DCT\ f(t,x)\}(s,w)$ exists,

$$\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}\{2DCT\ f(t,x)\}(s,w)=<f(t,x),\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}K_{f_1}(t,s)K_{f_2}(x,w)>$$

we prove the result $n=1$, the general result following by induction.

For fixed $s_j\ne 0$ choose two concentric circles C and C^1 with centre s_j and radii r and r_1 respectively such that $0<r<r_1<|s_j|$.

Let Δs_j be a complex increment satisfying $0<|\Delta s_j|<r$. Also for fixed $w_j\ne 0$.*Again* choose two concentric circles C and C_1 with centre w_j and radii r' and r_1' respectively such that $0<r'<r_1'<|w_j|$.

Let Δw_j be a complex increment satisfying $0<|\Delta w_j|<r'$

Consider,

$$\frac{(2DCT)(s_j+\Delta s_j,w_j)-(2DCT)(s_j,w_j)}{\Delta s_j}\frac{(2DCT)(s_j,w_j+\Delta w_j)-(2DCT)(s_j,w_j)}{\Delta w_j}$$
$$-<f(t,x),\frac{\partial}{\partial s_j}\frac{\partial}{\partial w_j}K_{f_1}(t,s)K_{f_2}(x,w)>$$

$$=<f(t,x),\Psi\Delta s_j(t)\Psi\Delta w_j(x)>\ldots\ldots..(5)$$

where $\Psi\Delta s_j(t)\Delta w_j(x)=\frac{1}{\Delta s_j}\left[K_{f_1}(t,s_1,s_2\cdots\cdots s_j+\Delta s_j\cdots s_n)-K_{f_1}(t,s)\right]$

$$\frac{1}{\Delta w_j}\left[K_{f_2}(x,w_1,w_2\cdots\cdots w_j+\Delta w_j\cdots w_n)-K_{f_2}(x,w)\right]-\frac{\partial^n}{\partial s_j^n}\frac{\partial^n}{\partial w_j^n}K_{f_1}(t,s)K_{f_2}(x,w)>$$

For any fixed $(t,x)\in R^n$ and any fixed integer.

$k=(k_1,k_2\cdots\cdots k_n)\in N_0^n$ and $l=(l_1,l_2\cdots\cdots l_n)\in N_0^n$

$D_t^kD_x^lK_{f_1}(t,s)K_{f_2}(x,w)$ is analytic inside and on C' and C_1'.

We have, by Cauchy integral formula.

$$D_t^k D_x^l\ \Psi\Delta s_j \Delta w_j(t,x) = \frac{1}{4\pi^2 i^2} D_t^k D_x^l K_{f_1}(t,s) K_{f_2}(x,w) \int_{c'}\int_{c_1'} \left(\frac{1}{\Delta s_j}\left(\frac{1}{z-s_j-\Delta s_j} - \frac{1}{z-s_j}\right) - \frac{1}{(z-s_j)^2}\right)$$

where,

$$\left(\frac{1}{\Delta w_j}\left(\frac{1}{y-w_j-\Delta w_j} - \frac{1}{y-w_j}\right) - \frac{1}{(y-w_j)^2}\right) dzdy$$

$\overline{s}$ = (s_1………s_{j-1}, z, s_{j+1}………s_n) *and* $\overline{w}$ = (w_1………w_{j-1}, y,w_{j+1}………w_n).

$$= \frac{\Delta s_j \Delta w_j}{-4\pi^2} \int_{c'}\int_{c_1'} \frac{D_t^k D_x^l\ K_{f_1}(t,\overline{s}) K_{f_2}(x,\overline{w})}{(z-s_j-\Delta s_j)\,(z-s_j)^2 (y-w_j-\Delta w_j)\,(y-w_j)^2}\, dzdy$$

But for all $z \in C'$ and $y \in C_1'$ and (t,x) restricted to a compact subset of R^n,

$D_t^k\ D_x^l K_{f_1}(t,s) K_{f_2}(x,w)$ is bounded by constantQ.

$$\left|D_t^k\ D_x^l \Psi\Delta s_j \Delta w_j(t,x)\right| \le \frac{|\Delta s_j||\Delta w_j|}{4\pi^2} \int_{c'}\int_{c_1'} \frac{Q}{(r_1-r)\,(r_1)(r_1'-r)\,(r_1')} |dz||dy|$$

$$\le \frac{|\Delta s_j||\Delta w_j|}{4\pi^2} \frac{Q}{(r_1-r)\,(r_1)(r_1'-r)\,(r_1')}$$

Thus as $|\Delta s_j| \to 0$, and $|\Delta w_j| \to 0$, $D_t^k\ D_x^l \Psi\Delta s_j \Delta w_j(t,x)$ tends to zero. Uniformly on the compact subset of R^n.Therefore it follows that $\Psi\Delta s_j \Delta w_j(t,x)$ converges in *E*(R^n) to zero since $f \in E^1$, we concluded (5) tends to zero. Therefore $\{2DCT\ f(t,x)\}(s,w)$ is differentiable with respective s_j and w_j. But this is true for all i, *j=1,2,……n .Hence*

$\{2DCTf(t,x)\}(s,w)$ is analytic on C^n and,

$$D_s^k D_w^l \{2DCT\ f(t,x)\}(s,w) = < f(t,x), D_s^k D_w^l\ K_{f_1}(t,s) K_{f_2}(x,w) >$$

IV. Properties of kernel:

If $\{2DCT\ f(t,x)\}(s,w) = \frac{1}{\sqrt{2\pi ib}} \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\left(\frac{s}{b}t\right)} e^{-i\left(\frac{w}{b}\right)x} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f(t,x)\,dxdt$

is definition two dimensional canonical transform of $f(t,x)$

Where,

$$k_{f_1}(t,s), k_{f_2}(x,w) = \frac{1}{\sqrt{2\pi ib}} \frac{1}{\sqrt{2\pi ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{\frac{i}{2}\left(\frac{d}{b}\right)w^2} e^{-i\left(\frac{s}{b}\right)t} e^{-i\left(\frac{w}{b}x\right)} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} \quad \text{when } b \neq 0$$

$$= \sqrt{d}\ e^{\frac{i}{2}\left(cds^2\right)} \delta(t-ds) \sqrt{d}\ e^{\frac{i}{2}\left(cdw^2\right)} \delta(x-dw) \quad \text{when b= 0}$$

kernel of 2D canonical transform satisfied following property

4.1) $k_{f_1}(t,s) k_{f_2}(x,w) = k_{f_1}(t,-s) k_{f_2}(x,w)$

4.2) $k_{f_1}(-t,s) k_{f_2}(-x,w) = k_{f_1}(t,-s) k_{f_2}(x,-w)$

4.3) $k_{f_1}(t,s) k_{f_2}(x,w) = k_{f_1}(s,t) k_{f_2}(w,x)$ if a=b

4.4) $k_{f_1}(t,s) k_{f_2}(x,w) = k_{f_1}(s,t) k_{f_2}(w,x)$ if a $\neq b$

Five properties of kernel, stated above are simple to prove, hence the proof omitted.

V. Conclusion:

The two-dimensional canonical transform is generalized in the distributional sense. Its inversion and Analyticity theorem is proved. Some properties of kernel are discussed. It can be used optical system analysis.

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